

ON SUPREMA OF AUTOCONVOLUTIONS WITH AN APPLICATION TO SIDON SETS

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ABSTRACT. Let f be a nonnegative function supported on the interval $(-1/4, 1/4)$ satisfying $\int_{-1/4}^{1/4} f(x)dx = 1$. The best constant in the inequality

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} f(t)f(x-t)dt \geq c$$

is of importance in additive combinatorics, where it appears in the asymptotic behavior of Sidon sets. The currently best bounds are $1.2748 \leq c \leq 1.5098$ and the upper bound is suspected to be almost tight. Using A_n to denote the compact set of all vectors $\mathbf{a} \in \mathbb{R}^{2n}$ having nonnegative entries summing up to $4n$, we prove

$$\min_{\mathbf{a} \in A_n} \max_{2 \leq \ell \leq 4n} \max_{-2n \leq k \leq 2n-\ell} \frac{1}{4n\ell} \sum_{k \leq i+j \leq k+\ell-2} a_i a_j \leq c$$

For a fixed n this is an explicit optimization problem over a compact set: numerical experiments suggests that this improves the currently best lower bound already for $n = 6$.

1. INTRODUCTION

1.1. **Sidon sets.** A subset $A \subset \{1, 2, \dots, N\}$ is called g -Sidon if

$$|\{(a, b) \in A \times A : a + b = m\}| \leq g$$

for every m . The obvious question is: how large can g -Sidon sets possibly be? Denote the answer by

$$\beta_g(n) := \max_{\substack{A \subset \{1, 2, \dots, N\} \\ A \text{ is } g\text{-Sidon}}} |A|.$$

There has been a lot of research activity on bounding these quantities: we only deal with the asymptotic case as g becomes large. Cilleruelo, Ruzsa & Vinuesa [1] have recently shown that the maximal cardinality of a g -Sidon set satisfies

$$\underline{\sigma(g)}\sqrt{gn}(1 - o(1)) \leq \beta_g(n) \leq \overline{\sigma(g)}\sqrt{gn}(1 + o(1)),$$

where the $o(1)$ is with respect to n and

$$\lim_{g \rightarrow \infty} \underline{\sigma(g)} = \sigma = \lim_{g \rightarrow \infty} \overline{\sigma(g)}$$

for some universal constant σ . Many papers are concerned with (or contain results implying) bounds for σ . As for the upper bound, we have

$\sigma \leq \sqrt{2}$	trivial
≤ 1.318	Cilleruelo, Ruzsa & Trujillo [2]
≤ 1.303	Green [4]
≤ 1.30	Martin & O'Bryant [6]
≤ 1.264	Yu [11]
≤ 1.258	Martin & O'Bryant [7]

Work on the lower bound has been carried out by Cilleruelo, Ruzsa & Trujillo [2], Kolountzakis [5], Martin & O'Bryant [6] and Cilleruelo & Vinuesa [3]. The current state of the art is

$$1.1509... \leq \sigma \leq 1.2525...$$

with both bounds coming from Matolcsi & Vinuesa [9]. The lower bound is conjectured to be almost tight. Our contribution will be the description of a method that (assuming sufficient computational power to carry out a certain optimization problem) seems to be able to show at least $\sigma \leq 1.25$ (and probably even much better results).

1.2. An equivalent continuous problem. As proven by Cilleruelo, Ruzsa & Vinuesa [1], the constant σ has an alternative representation as the solution of a continuous problem involving the autoconvolution $f * f$ which was first considered by Schinzel & Schmidt [10]. Throughout this paper, we will use

$$(f * g)(x) = \int_{\mathbb{R}} f(t)g(x-t)dt$$

to denote the convolution. Consider all nonnegative functions f supported on $[0, 1]$ satisfying

$$\|f * f\|_{L^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} f(t)f(x-t)dt \leq 1,$$

then we have

$$\int_{\mathbb{R}} f(x)dx \leq \sigma$$

and σ is the optimal constant in that inequality. Henceforth, we will only be concerned with the continuous problem stated in an equivalent form (which is the form considered by Matolcsi & Vinuesa [9]): let f be a nonnegative function supported on the interval $[-1/4, 1/4]$ with normalized mean

$$\int_{-1/4}^{1/4} f(x)dx = 1.$$

What bounds can be proven on the optimal constant $c > 0$ in the inequality

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} f(t)f(x-t)dt \geq c.$$

Scaling yields the relation

$$\sigma = \sqrt{\frac{2}{c}}.$$

The scaling relation combined with the results of Matolcsi & Vinuesa [9] implies

$$1.2748 \leq c \leq 1.5098,$$

where the upper bound disproves the conjecture $c = \pi/2$ due to Schinzel & Schmidt [10]. The upper bound comes from an explicit example; the lower bound is established using Fourier methods and uses earlier arguments of Martin & O'Bryant [8] and Yu [11]. While it is possible to try to improve the upper bounds for c by constructing further explicit examples (as was done in [9]), this approach seems to be exhausted: indeed, Matolcsi & Vinuesa write that $c \sim 1.5$ seems reasonable "unless there exists some hidden 'magical' number theoretical construction yielding a much smaller value (the possibility of which is by no means excluded)". Matolcsi & Vinuesa claim the theoretical limit of their approach for the lower bound to be at 1.276. Clearly, new ideas on how to obtain lower bounds are needed.

1.3. The result. The purpose of this paper is to present such a new idea. We regard convolution as a bilinear operator that respects spatial decompositions and the L^1 -norm. This viewpoint allows us to construct a sequence of explicitly computable real numbers $(a_n)_{n \in \mathbb{N}}$ of lower bounds for c : every element a_n of the sequence is given as the solution of a minimization problem over the $2n$ -dimensional simplex $A_n \subset \mathbb{R}^{2n}$ written as

$$A_n = \left\{ (a_{-n}, a_{-n+1}, \dots, a_{n-1}) \in (\mathbb{R}_+)^{2n} : \sum_{i=-n}^{n-1} a_i = 4n \right\}.$$

The statement reads as follows.

Theorem. For any $n \in \mathbb{N}$ let

$$a_n := \min_{a \in A_n} \max_{2 \leq \ell \leq 4n} \max_{-2n \leq k \leq 2n-\ell} \frac{1}{4n\ell} \sum_{k \leq i+j \leq k+\ell-2} a_i a_j,$$

where $k, \ell \in \mathbb{Z}$. Then

$$c \geq a_n.$$

We hope that this result will stimulate further research: in particular, it is not clear to us whether the underlying idea could be used in a way that is more amenable to numerical methods or which computational methods would prove most effective.

1.4. Numerical results. For lack of trust in our ability to safely do high-precision numerics and the fact that even the smallest computer cluster would (and hopefully will) dwarf any of our potential efforts, we have only estimated the first 5 values from above by testing examples: it is easy to see that $a_1 = 1$. In the other cases, we created 100.000 random vectors by choosing all entries uniformly from $[0,1]$, normalizing the entries and finding the one yielding the smallest value.

n	1	2	3	4	5
a_n	= 1	≤ 1.111	≤ 1.181	≤ 1.234	≤ 1.28

FIGURE 1. The best upper bounds for a_n found by Monte-Carlo methods.

Assuming both these bounds to be close to the truth and the pattern to continue, it seems natural to expect $a_6 \geq 1.28$. We believe an accurate computation of the sequence at least up to $n \leq 10$ to be well within current (though not the authors') possibilities. Of course, given a result of this type, the natural question is whether

$$\sup_{n \in \mathbb{N}} a_n = c?$$

While the author is inclined to believe that this is the case, he was not able to prove it. However, even a negative answer to that question would provide further insight: as will be seen later in the proof, each vector in A_n corresponds to an associated step function and step functions corresponding to the minimizing vector must induce very peculiar autoconvolutions – in a certain sense, this would correspond to the "magical number theoretical constructions" hinted above.

1.5. Some intuition. Before proving the Theorem, we describe the underlying idea. There is a trivial proof of $c \geq 1$: since f is supported on $(-1/4, 1/4)$, the function $f * f$ must be supported on $(-1/2, 1/2)$. Furthermore, by Fubini, we have that

$$\int_{\mathbb{R}} (f * f)(x) dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) f(x-t) dt dx = \left(\int_{\mathbb{R}} f(x) dx \right)^2 = 1$$

and thus, trivially

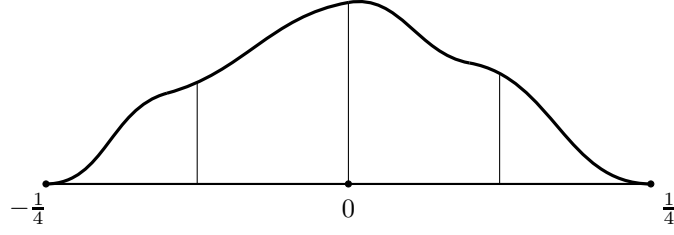
$$1 = \|f * f\|_{L^1(-1/2, 1/2)} \leq \|f * f\|_{L^\infty(-1/2, 1/2)}.$$

This argument inspired our approach: instead of trying to show that $\|f * f\|_{L^\infty}$ has to be large by showing that it has to be large in a single point, we show that there has to exist an interval $I \subset (-1/2, 1/2)$ such that $\|f * f\|_{L^1(I)}$ is large and then use

$$\frac{1}{|I|} \|f * f\|_{L^1(I)} \leq \|f * f\|_{L^\infty(I)} \leq \|f * f\|_{L^\infty(\mathbb{R})}.$$

Let us describe the argument in greater detail (corresponding to the case $n = 2$ in the Theorem): we decompose the interval $(-1/4, 1/4)$ into four intervals I_1, I_2, I_3, I_4 of equal size in the canonical way, i.e.

$$I_j = \left(\frac{-3+j}{8}, \frac{-2+j}{8} \right) \quad \text{for } j = 1, 2, 3, 4.$$

FIGURE 2. A nonnegative function supported on $(-1/4, 1/4)$.

Denoting the characteristic function of an interval I by χ_I we define

$$f_j = f \chi_{I_j}$$

and use the linearity of the convolution to write

$$\begin{aligned} f * f &= \left(\sum_{j=1}^4 f_j \right) * \left(\sum_{j=1}^4 f_j \right) \\ &= \sum_{j,k=1}^4 f_j * f_k \end{aligned}$$

It is easy to see that $f_j * f_k$ is supported on $I_j + I_k$, where $+$ is interpreted as the Minkowski sum of sets. This property allows now to deduce precisely where the support of each term is. We have, additionally, by Fubini, that

$$\int_{\mathbb{R}} (f_j * f_k)(x) dx = \left(\int_{\mathbb{R}} f_j(x) dx \right) \left(\int_{\mathbb{R}} f_k(x) dx \right).$$

This allows now to deduce that we have, for example, the pointwise inequality

$$(f_1 * f_1)(x) + (f_1 * f_2)(x) + (f_2 * f_1)(x) \leq (f * f)(x) \quad \text{on the interval } (-1/2, -1/8).$$

Integrating the inequality on both sides yields

$$\left(\int_{\mathbb{R}} f_1(x) dx \right)^2 + 2 \left(\int_{\mathbb{R}} f_1(x) dx \right) \left(\int_{\mathbb{R}} f_2(x) dx \right) \leq \int_{-1/2}^{-1/8} (f * f)(x) dx \leq \frac{3}{8} \|f * f\|_{L^\infty(\mathbb{R})}.$$

This corresponds to certain restrictions on the distribution of the L^1 -mass of the function over the intervals assuming $\|f * f\|_{L^\infty(\mathbb{R})}$ to be small, or, arguing conversely, shows that any function with $\|f * f\|_{L^\infty(\mathbb{R})}$ small must induce a partition of its L^1 -mass that satisfies the particular inequality. The statement then follows by repeating the argument over several different intervals.

2. PROOF OF THE THEOREM

Proof. Let $\varepsilon > 0$ be arbitrary and let f be a nonnegative function supported on the interval $(-1/4, 1/4)$ with normalized mean

$$\int_{-1/4}^{1/4} f(x) dx = 1$$

such that

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} f(t) f(x-t) dt \leq c + \varepsilon.$$

Consider the decomposition of the interval into $2n$ equally sized intervals

$$\left(-\frac{1}{4}, \frac{1}{4} \right) = \bigcup_{j=-n}^{n-1} I_j \quad \text{where} \quad I_j = \left(\frac{j}{4n}, \frac{j+1}{4n} \right).$$

We denote the restriction of f to the interval I_j by f_j and define the average on that region by

$$a_j = \frac{1}{|I_j|} \int_{I_j} f(x) dx = 4n \int_{\mathbb{R}} f_j(x) dx.$$

Trivially, since $f \geq 0$, we have $a_j \geq 0$ as well as

$$\frac{1}{4n} \sum_{j=-n}^{n-1} a_j = \int_{-1/4}^{1/4} f(x) dx = 1$$

and thus $(a_{-n}, a_{-n+1}, \dots, a_{n-1}) \in A_n$. Let $2 \leq \ell \leq 4n$ be an arbitrary integer and let $-2n \leq k \leq 2n - \ell$ be another arbitrary integer. A simple computation yields

$$\begin{aligned} \frac{1}{4n\ell} \sum_{k \leq i+j \leq k+\ell-2} a_i a_j &= \frac{4n}{\ell} \sum_{k \leq i+j \leq k+\ell-2} \frac{a_i}{4n} \frac{a_j}{4n} \\ &= \frac{4n}{\ell} \sum_{k \leq i+j \leq k+\ell-2} \left(\int_{I_i} f(x) dx \right) \left(\int_{I_j} f(x) dx \right) \\ &= \frac{4n}{\ell} \sum_{k \leq i+j \leq k+\ell-2} \int_{\mathbb{R}} (f_i * f_j)(x) dx \\ &\leq \underbrace{\frac{4n}{\ell} \int_{\frac{k}{4n}}^{\frac{k+\ell}{4n}} (f * f)(x) dx}_{(\diamond)} \\ &\leq \frac{4n}{\ell} \int_{\frac{k}{4n}}^{\frac{k+\ell}{4n}} \|f * f\|_{L^\infty(\mathbb{R})} dx \\ &= \|f * f\|_{L^\infty} \leq c + \varepsilon, \end{aligned}$$

where (\diamond) is the only nonobvious step, which follows from the fact that $k \leq i + j \leq k + \ell - 2$ implies that

$$I_i + I_j \subseteq \left(\frac{k}{4n}, \frac{k+\ell}{4n} \right).$$

Since ε was arbitrary, the compactness of A_n now implies

$$\min_{a \in A_n} \max_{2 \leq \ell \leq 4n} \max_{-2n \leq k \leq 2n - \ell} \frac{4n}{\ell} \sum_{k \leq i+j \leq k+\ell} a_i a_j \leq c.$$

□

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